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THE LONG-PERIOD MOTION OF THE TROJANS, WITH SPECIAL ATTENTION TO THE THEORY OF THUERING

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SUMMARY

In 1930, B. Thuering published an approximative theory of long-period motion of the Trojans as a restricted three-body problem. With this theory we can approximate the periodic orbits for all amplitudes in such a favorable manner that the remaining deviations are at most of the order of the mass of Jupiter ($< 10^{-3}$). Thuering's solution provides a starting point for an exact theory of the plane long-period Trojan orbits according to the method of the variation of constants. Special attention is devoted to the borderline case, in which the periodic orbits around L_4 and L_5 overlap. This boundary orbit and its adjacent orbits run into the infinitesimal Charlier orbits around L_3 — more specifically, into those of the hyperbola type.

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THE LONG-PERIOD MOTION OF THE TROJANS, WITH SPECIAL ATTENTION TO THE THEORY OF THUERING*

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INTRODUCTION

The first comprehensive attempt at devising a theory of periodic motions around the Lagrange libration centers L_1, L_2, $\cdots$, L_5 in the restricted problem was undertaken by C.V.L. Charlier around the turn of the century. This was a short time before the discovery, in 1906, of the first *Trojan* (the planet 588 Achilles) lent practical meaning to this problem which formerly had been of interest only to the theoretician. Charlier's theory merely considered infinitesimal orbits around the libration centers, i.e., orbits whose distances from the libration center remain so small that their squares can be neglected. Charlier showed that there are two families of infinitesimal periodic orbits around L_4 and L_5 (which form an equilateral triangle with the two finite masses sun and Jupiter) in the rotating coordinate system (in which the sun and Jupiter hold fixed positions on the x-axis); each of these two families of infinitesimal period orbits consists of a group of concentric, coaxial, and similar ellipses. In each family, the rotation time of all members is equal. The short-period orbits of one family have rotation periods which are only a little longer than those of Jupiter (T_0 = 11.86 years) and which converge on this value, if we allow the Jupiter mass to decrease toward zero. Their ellipticity coefficient $b:a$ deviates only slightly from the boundary value 1:2, toward which it tends when m approaches 0. The minor axis is pointed at the sun (with minor deviations which also disappear when m goes to 0); the major axis thus lies roughly along the tangent of the Jupiter orbit at L_4 (or L_5 respectively).

This family of short-period libration orbits has a very simple meaning which was apparently accorded little attention in the literature. Let us consider a planet moving around the sun in an (undisturbed) elliptical orbit whose semimajor axis and rotation period are equal to those of Jupiter $(a = 1,\ T = T_0)$ and whose eccentricity e is so small that magnitudes of the order of e^2 can be neglected. Thus the orbit of this body in the rotating system is a small ellipse around a fixed point of a circle of unit radius, its ellipticity coefficient is 1:2, its major axis is tangent to the

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circle of unit radius, and its rotation period is T_0. The family of the short-period libration orbits around L_4 or L_5 thus converges if we allow the disturbing mass of Jupiter to decrease toward zero toward a group of Kepler ellipses with the same rotation time, the same perihelion longitude and the minor eccentricity e, which serves as group parameter.

The orbits of the second family — called Trojan orbits — are concentric to and coaxial with the former. If m is the mass of Jupiter measured in units of solar mass, we have the following approximation: $b:a = \sqrt{3m}$, $T = T_0\sqrt{4/27m}$ as the common ellipticity coefficient and the common rotation time of the orbit ellipses. For $m = 1/1{,}047$, we get $b:a = 1:18.7$ $T = 148$ years.

Attempts to extend the theory of the Trojan orbits to noninfinitesimal areas around L_4, L_5 were soon undertaken, such as the work by H. C. Plummer, E. W. Brown, and others (References 1 and 2). These investigations, confined to the consideration of the squares of the planetoid coordinates in relation to the center of libration, revealed a deformation of the orbit ellipses as the distance from the center increased: (1) a slight shift of the center on the circle of unit radius, that is, in the direction away from Jupiter; (2) the symmetry line of the orbit hugs the circle of unit radius; (3) the curvature of the orbit in the greatest elongations on the side facing away from Jupiter is stronger than on the side near Jupiter. On the other hand, the rotation period – considering first-order terms — remains unchanged and begins to increase gradually only as the libration amplitudes grows and as second and higher powers of the coordinates are considered.

AN IMPROVED TROJAN ORBIT

The first attempt at obtaining a clearer picture of the Trojan orbit of arbitrary amplitude width was made as early as 1911 by E. W. Brown (Reference 1), who introduced the polar coordinates r, ϕ (where r is the distance from sun, and ϕ the difference of lengths of the planetoid and Jupiter). And he was able to show that $r - 1$ and $\dot{\phi}$ are, at most, of the order of $\sqrt{m}$, while $\dot{r}$ and $\ddot{\phi}$ are at most of the order of m, etc. Thus the orbit lies in the vicinity of a circle of unit radius (Figure 1); and if we use α to designate the average libration amplitude (the average distance of the Trojan from the libration center in the maximum elongations), then $\alpha\sqrt{m}$ is the order of magnitude of the maximum distance of the Trojan from the circle of unit radius. In addition, Brown succeeded in estimating the distances of the Trojan from the libration center in the elongations; in particular he found that, if $\phi_0 = 60°$ is the length of L_4 (related to the longitude of Jupiter), the longitude of the longest elongation on the side facing away from Jupiter approaches 180°, while that of the elongation near Jupiter decreases to about 24°. We thus have a boundary orbit in the family of the Trojan orbits around L_4 which, in the elongation facing away from Jupiter, reaches to the opposite point of Jupiter relative to the sun and there meets the corresponding boundary orbit around L_5, although the elongation of the boundary orbits on the side near Jupiter remains separated from Jupiter by a longitude difference of about 24°. The elongations of the boundary orbits of L_4 therefore are 120° and 36°, respectively, and the deviations of these orbits from the circle of unit radius are of the order of $\sqrt{m} \approx 1/30$.

The motion theory of the real Trojans, of which nine around L_4 and five around L_5 have become known thus far, has been worked out by various authors, of whom we shall mention only E. W. Brown

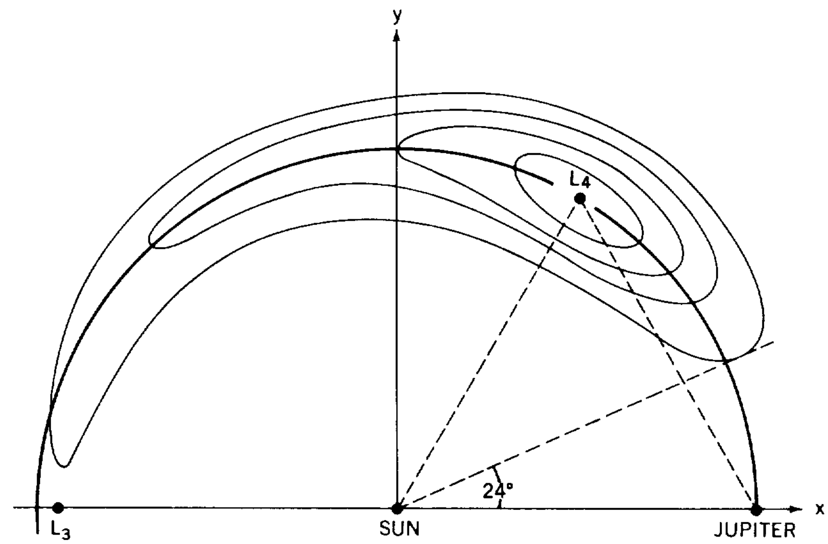

Figure 1—Family of periodic Trojan orbits. The deviation of the curves from the unit circle is exaggerated.

(Reference 3) and A. Wilkens (References 4 through 7). These theories concern the spatial movement of these planets, including all perturbations due to Jupiter and other planets, especially Saturn. These orbits are not periodic. In the plane restricted problem – a problem of the fourth order represented by two differential equations of the second order, every libration around L_4 (or L_5) is composed of two independent motions which have the characteristics of the two Charlier families (expanded to noninfinitesimal regions). In this problem we have periodic Trojan orbits with rotation times (equal to or greater than 148[a]) obtained by setting the initial conditions so that the short-period component vanishes. Thus these periodic Trojan orbits are solutions of a second-order system.

THUERING'S WORK ON LONG-PERIOD LIBRATIONS

Owing to Thuering's work (Reference 8), we now have an approximated theory of these long-period librations. Simply and clearly, this theory gives intermediate orbits, whose deviation from the exact-period orbits does not exceed the order of m even in the boundary case. Thus these orbits can be used as approximations for an exact theory and therefore provide a very clear picture of the shape of the orbits and the form of motion within them.

In a plane coordinate system, x_1 , y_1 are the rectangular, heliocentric coordinates of Jupiter which moves around the sun in a circular orbit (radius $a_1 = 1$). Thus, for Jupiter, we have:

$r_1 = 1$, the radius vector = 5.20 A.E.,

$\lambda_1 = n_1 (t - t_0)$, the mean length,

$l_1 = \lambda_1$, the true length;

and we may let $n_1 = \sqrt{1 + m}$ for the mean motion in the time unit, if we set this time unit as equal to $(5.20)^{3/2}/k$ average days. We also let x, y be the rectangular coordinates and r, l the polar coordinates of the Trojan.

As elements of the osculatory orbit of the Trojan for any time t, we introduce:

a = major semiaxis,

$e = \sin\phi$ = numerical eccentricity,

ω = length of the perihelion.

Furthermore, let M be the mean anomaly, so that $l = \lambda + 2e\sin M$ + terms of higher order in e, where $\lambda = n(t - t_0) = a^{-3/2}(t - t_0)$ is the mean longitude of the planetoid.

Using the canonic elements of Poincaré,

$$L = \sqrt{a}, \qquad \lambda = M + \omega,$$

$$p = 2\sqrt{a}\sin^2(\phi/2), \qquad q = -\omega,$$

we obtain the following differential equations for the motion:

$$\frac{dL}{dt} = \frac{\partial F}{\partial \lambda}, \qquad \frac{dp}{dt} = \frac{\partial F}{\partial q},$$

$$\frac{d\lambda}{dt} = -\frac{\partial F}{dL}, \qquad \frac{dq}{dt} = -\frac{\partial F}{\partial p}, \tag{1}$$

where the Hamilton function

$$F = \frac{1}{2a} + m\left[\frac{1}{s} - \left(xx_1 + yy_1\right)\right].$$

From Figure 2 we have

$$s^2 = 1 + r^2 - 2r\cos\Psi,$$

$$xx_1 + yy_1 = r\cos\Psi,$$

$$\begin{aligned}\Psi &= l - l_1 \\ &= \lambda - \lambda_1 + 2e\sin M + \cdots \\ &= \alpha + 2e\sin M + \cdots.\end{aligned}$$

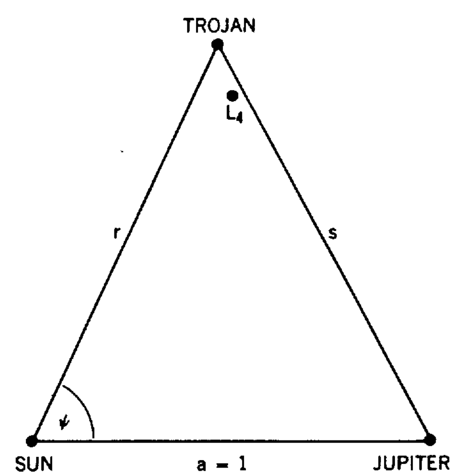

Figure 2–The triangle sun-Jupiter-Trojan.

Now, in the osculatory orbit Brown showed that

$$\dot{r} = \frac{e\sin v}{\sqrt{a\left(1 - e^2\right)}},$$

where $\dot{r}$ is, at most, of the order of m (for $v = 90°$). From this it follows that e is also the order of m. Therefore we shall develop F into a power series of e,

$$F = F_0 + eF_1 + e^2 F_2 + \cdots ,$$

where F_0 is free of the eccentricity e. However, $F = F(r, \psi)$, for, if we set up $a = a_1(1+\rho) = 1+\rho$, $r = a(1 - e\cos E)$, $\psi = \alpha + 2e\cos M + \cdots$,

$$F = F(\rho, \alpha, e\cos E, e\cos M, \cdots) .$$

And for $e = 0$ we have

$$\begin{aligned} F_0 &= F_0(\rho, \lambda) \\ &= F_0(\rho, \alpha) \\ &= \frac{1}{2(1+\rho)} + m\left[\frac{1}{\sqrt{2(1+\rho)(1-\cos\alpha) + \rho^2}} - (1+\rho)\cos\alpha\right] . \end{aligned} \tag{2}$$

The differential Equations 1 are limited to the second order system, if we neglect magnitudes of the order of e, and of the order of m:

$$\frac{dL}{dt} = \frac{\partial F_0}{\partial \lambda}, \qquad \frac{d\lambda}{dt} = -\frac{\partial F_0}{\partial L},$$

or if we set up

$$\begin{aligned} L &= \sqrt{1+\rho} , \\ \lambda &= \lambda_1 + \alpha \\ &= \alpha + \sqrt{1+m}\,(t - t_0) , \end{aligned}$$

they are limited to

$$\begin{aligned} \frac{d\rho}{dt} &= 2\sqrt{1+\rho}\,\frac{\partial F_0}{\partial \alpha} , \\ -\frac{d\alpha}{dt} &= \sqrt{1+m} + 2\sqrt{1+\rho}\,\frac{\partial F_0}{\partial \rho} . \end{aligned} \tag{3}$$

The remaining two equations of Equation 1 are now superfluous, since p vanishes and $q = -\omega$ becomes meaningless when $e = 0$.

If in Equation 2 we now ignore all terms of the order m^2 (noting that ρ^2 is of the order of m), we get

$$F_0 = \frac{1}{2(1+\rho)} + m\left[\frac{1-\frac{\rho}{2}}{2\sin\frac{\alpha}{2}} - (1+\rho)\cos\alpha\right]$$

and

$$\frac{\partial F_0}{\partial\alpha} = m\left[\frac{\frac{\rho}{2}-1}{4\sin^2\frac{\alpha}{2}}\cos\frac{\alpha}{2} + (1+\rho)\sin\alpha\right]$$

$$= m\sin\alpha\left[1 - \frac{1}{8\sin^3\frac{\alpha}{2}} + \rho\left(1 + \frac{1}{16\sin^3\frac{\alpha}{2}}\right)\right],$$

$$\frac{\partial F_0}{\partial\rho} = -\frac{1}{2(1+\rho)^2} - m\left(\frac{1}{4\sin\frac{\alpha}{2}} + \cos\alpha\right).$$

From the preceding two equations and Equations 3, we get

$$\frac{d\rho}{dt} = 2m\sqrt{1+\rho}\,\sin\alpha\left[1 - \frac{1}{8\sin^3\frac{\alpha}{2}} + \rho\left(1 + \frac{1}{16\sin^3\frac{\alpha}{2}}\right)\right] \tag{4}$$

and

$$\frac{d\alpha}{dt} = (1+\rho)^{-3/2} - \sqrt{1+m} + 2m\sqrt{1+\rho}\left(\frac{1}{4\sin\frac{\alpha}{2}} + \cos\alpha\right) \tag{5}$$

If we differentiate Equation 5 again with respect to time and replace $d\rho/dt$ with Equation 4 and then ignore all terms of the order of m^2, we obtain

$$\frac{d^2\alpha}{dt^2} = -3m(1+\rho)^{-2}\sin\alpha\left[1 - \frac{1}{8\sin^3\frac{\alpha}{2}} + \rho\left(1 + \frac{1}{16\sin^3\frac{\alpha}{2}}\right)\right]$$

$$-2m\sqrt{1+\rho}\,\sin\alpha\left(1 + \frac{1}{16\sin^3\frac{\alpha}{2}}\right)\frac{d\alpha}{dt}. \tag{6}$$

But according to Equation 5, we have

$$\frac{d\alpha}{dt} = -\frac{3}{2}\rho + \text{terms of the order of } m, \text{ and also, } \rho^2, \tag{7}$$

ρ and $d\alpha/dt$ thus, as we said before, are of the same order $\sqrt{m}$. If we now ignore in Equation 6 all terms of the order m^2, $m\rho^2$, $m\rho(d\alpha/dt)$, we may, from Equation 7, let

$$\rho = -\frac{2}{3}\frac{d\alpha}{dt} \tag{8}$$

and instead of the two differential equations of the first order (Equations 4 and 5), we obtain one differential equation of the second order for $\alpha(t)$:

$$\frac{d^2\alpha}{dt^2} + f(\alpha)\left(\frac{d\alpha}{dt} + \frac{3}{4}\right) = 0$$

where

$$f(\alpha) = 4m \sin\alpha \left(1 - \frac{1}{8\sin^3\frac{\alpha}{2}}\right) \tag{9}$$

The solution of Equation 9 also determinates ρ from Equation 8 (except for terms of the order m).

The integration of Equation 9 can be accomplished in the following elegant manner. We have a known function of α,

$$\phi(\alpha) = -\int f(\alpha)\, d\alpha = 4m\left(\cos\alpha - \frac{1}{2\sin\frac{\alpha}{2}}\right).$$

Now, if we let

$$\frac{d\alpha}{dt} = \phi(\alpha) + \gamma(\alpha), \tag{10}$$

it suffices to determine $\gamma(\alpha)$, for afterwards $\rho(t)$ is obtained from Equation 8 and $\alpha(t)$ by squaring Equation 10. From Equation 10, it follows that

$$\begin{aligned}
\frac{d^2\alpha}{dt^2} &= \frac{d\phi}{dt} + \frac{d\gamma}{dt} \\
&= \left(\frac{d\phi}{d\alpha} + \frac{d\gamma}{d\alpha}\right)\frac{d\alpha}{dt} \\
&= \left[\frac{d\gamma}{d\alpha} - f(\alpha)\right]\frac{d\alpha}{dt}.
\end{aligned}$$

If we insert the above equation into Equation 9, we get

$$\frac{d\gamma}{d\alpha}\frac{d\alpha}{dt} + \frac{3}{4} f(\alpha) = 0$$

or

$$\frac{d\gamma}{d\alpha} = -\frac{3}{4}\frac{f(\alpha)}{\phi(\alpha) + \gamma(\alpha)} \quad . \tag{11}$$

If, for the moment, we set

$$\phi + \gamma = \frac{3}{4u} \quad , \tag{12}$$

we find that

$$\frac{d}{d\alpha}(\phi + \gamma) = -f(\alpha) + \frac{d\gamma}{d\alpha}$$

$$= -\frac{3}{4u^2}\frac{du}{d\alpha} \ ;$$

or, if from Equation 11, we set

$$\frac{d\gamma}{d\alpha} = -uf(\alpha) \ ,$$

we find that

$$\frac{du}{d\alpha} = \frac{4}{3} fu^2 (1 + u) \ ,$$

i.e.,

$$\frac{du}{u^2(1+u)} = \frac{4}{3} f(\alpha)\, d\alpha$$

$$= -\frac{4}{3} d\phi \quad .$$

This equation is integrable and, if ϕ_0 is an arbitrary constant, we obtain

$$\frac{4}{3}(\phi - \phi_0) = \frac{1}{u} + ln \frac{u}{1+u} \quad .$$

If, instead of u, we again introduce γ from Equation 12, we get

$$-\frac{4}{3}(\phi_0 + \gamma) = \ln \frac{1}{1 + \frac{4}{3}(\phi + \gamma)} ,$$

or

$$e^{-(4/3)(\phi_0+\gamma)}\left[1 + \frac{4}{3}(\phi+\gamma)\right] = 1 , \tag{13}$$

an equation which determines $\gamma = \gamma(\phi, \phi_0)$. Since from Equation 10, ϕ is of the order of m and γ is of the order $\sqrt{m}$, we can solve Equation 13 by using a rapidly converging power series. From Equation 13 and

$$y = \frac{4}{3}\frac{d\alpha}{dt} = \frac{4}{3}(\phi + \gamma),$$

$$z = \frac{4}{3}(\phi - \phi_0) ,$$

we have

$$e^{-y}(1 + y) = e^{-z} ,$$

i.e.,

$$1 - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{8}y^4 + \cdots = 1 - z + \frac{1}{2}z^2 - \cdots ; \tag{14}$$

and to a first approximation,

$$y^2 \approx 2z = \zeta^2 ,$$

where

$$\zeta = \pm\sqrt{2z} .$$

Therefore we can set up a power series of ζ for y and by comparing coefficients we obtain the following from Equation 14:

$$y = \pm\zeta - \frac{1}{3}\zeta^2 \pm \frac{1}{36}\zeta^3 - \frac{1}{135}\zeta^4 \pm \cdots$$

or, if we neglect terms in ζ^4 and higher terms (which are at least of the order of m^2) we get

$$y = \frac{4}{3}\frac{d\alpha}{dt}$$

$$\approx \pm \zeta \left(1 \mp \frac{1}{6}\zeta\right)^2 \quad .$$

We now have the approximation

$$\frac{d\alpha}{dt} = \pm\frac{3}{4}\zeta\left(1 \mp \frac{1}{6}\zeta\right)^2$$

$$= \pm\sqrt{\frac{3}{2}(\phi - \phi_0)}\left[1 \mp \frac{1}{3}\sqrt{\frac{2}{3}(\phi - \phi_0)}\right]^2 \quad , \tag{15}$$

where for real orbits

$$\phi = 4m\left(\cos\alpha - \frac{1}{2\sin\frac{\alpha}{2}}\right) \geqq \phi_0 \quad . \tag{16}$$

THE MOTION OF THE TROJANS

The motion takes place in such a manner that α, the longitudinal difference between Trojan and Jupiter, fluctuates between two limits $\alpha_1 \geqq \alpha_2$. These boundary points, which are return points between the clockwise and counterclockwise motions with respect to L_4, are given by $d\alpha/dt = 0$, (i.e., $\phi = \phi_0$). This equation, when $x = \sin(\alpha/2)$, can also be written in the algebraic form

$$x^3 - \frac{1}{2}x\left(1 - \frac{\phi_0}{4m}\right) + \frac{1}{4} = 0 \quad . \tag{17}$$

Equation 17 has, at most, two real roots in the range $0 < \alpha \leqq 180°$, that is: for $0 < x \leq 1$; for $\phi_0 = -2m$ the double root is $x = 1/2$, $\alpha = 60°$. For this value of the parameter ϕ_0, α is constant, therefore the planetoid remains constantly in L_4. For the boundary orbit, we have the maximum elongation $x = 1$ in the side away from Jupiter when $\alpha_1 = 180°$ (i.e., for $\phi_0 = -6m$). The corresponding second root of Equation 17 results from $\alpha_2 = 23.°9$. Figure 3 shows the position of the return points for different values of ϕ_0 between $-2m$ and $-6m$.

The maximum angular speed of the Trojan relative to L_4 occurs when $d^2\alpha/dt^2 = 0$, because, from Equation 9, $f(\alpha) = 0$, since $d\alpha/dt + 3/4$ is not equal to zero. But this condition is met for $\sin\alpha/2 = 1/2$ (i.e., $\alpha = 60°$). The angular speed thus reaches its maximum if the planetoid, seen from the sun, passes the libration center. When $\alpha = 60°$ we get $\phi = -2m$ from Equation 16 and hence, we get $\phi - \phi_0 = 4m$ for case of the boundary orbit ($\phi_0 = -6m$). If we insert this result in Equation 15

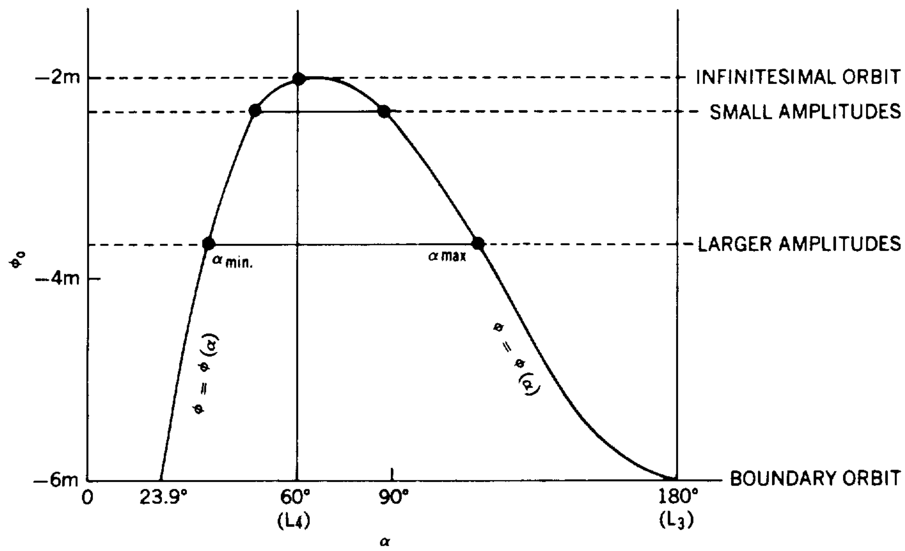

Figure 3—Limits of α, for Trojan orbits of different amplitude.

and if we consider that the time unit was so selected that $T_0 = 2\pi$, and if we use the mean day as the unit of time, we get for the maximum angular speed in the boundary orbit +23.″4 and -21.″9 daily, respectively.

Since ρ is proportional to $-\dot{\alpha}$, it follows that the motion occurs in the positive sense (counterclockwise) when the Trojan traverses that part of its orbit which lies within the orbit of Jupiter. The motion around L_4 (this also applies to L_5) is thus counterclockwise, and the angular speed is somewhat greater on the inside than on the outside part. Thuering gives a rougher approximation to the solution of Equation 14 and obtains $\dot{\alpha}_{max} = \pm$ 22.″6, as a medium value.

The rotation time in the periodic orbit can be obtained through integration of Equation 15. From

$$dt = \pm\frac{4}{3}\frac{d\alpha}{\zeta\left(1 \pm \frac{1}{6}\zeta\right)^2}, \quad \zeta = \sqrt{\frac{8}{3}(\phi - \phi_0)}, \quad \phi = \phi(\alpha)$$

we get

$$t - t_0 = \pm\frac{4}{3}\int_{\alpha_0}^{\alpha}\frac{d\alpha}{\zeta\left(1 \pm \frac{1}{6}\zeta\right)^2},$$

which means that the time for the inside orbit arc is

$$P_1 = \frac{4}{3}\int_{\alpha_{min}}^{\alpha_{max}}\frac{d\alpha}{\zeta\left(1 + \frac{1}{6}\zeta\right)^2};$$

and the time for the outside orbit arc is

$$P_2 = \frac{4}{3}\int_{\alpha_{min}}^{\alpha_{max}} \frac{d\alpha}{\zeta\left(1 - \frac{1}{6}\zeta\right)^2} .$$

Therefore the total rotation time is $P = P_1 + P_2$.

Since the motion of the Trojan is always very slow even in the boundary orbit, we find, on the basis of the Jacobi integral $y^2 = 2\Omega - C$, that the orbits are always near the Hill zero speed curves: $2\Omega - C = 0$ and approach these curves particularly closely at the return points. Every Trojan orbit is therefore enclosed between two neighboring Hill curves with slightly differing values of C. From this we can conclude that the boundary orbit, whose outermost limit is at $\alpha = 180°$, will run into that Hill curve which represents the boundary case between the isolated, bean-shaped curves around L_4 and L_5 and the horse-shoe-shaped curves which surround the two libration centers. This boundary curve has a double point in L_3, the libration center which is situated near the point of the x-axis $(x = -1, y = 0)$ opposite Jupiter, and has the abscissa $x = -1 + (7/12)m$ (except for the terms of the order of m^2).

CONCLUDING REMARKS

According to the Charlier theory we have two families of orbits around L_3, as well as around all other libration centers; one of these families is periodic and, in the infinitesimal range, consists of concentric, coaxial, and similar ellipses whose ellipticity ratio for $m \to 0$ is transformed into $b:a = 1:2$ and the rotation time becomes T_0. Like the family of the short-period orbits around L_4 and L_5, this family represents slightly eccentric Kepler ellipses which are little disturbed by Jupiter. The second family consists of nonperiodic orbits, which can be represented in the vicinity of L_3 by a group of concentric, coaxial, and similar hyperbolas or, rather, by two groups whose common asymptotes touch the above mentioned Hill boundary curves in L_3. These conditions are shown in Figure 4; there is no doubt that the pointed hyperbolas in the upper and lower sector of the figure belong to those Trojan orbits around L_4 and L_5, which in the boundary orbit approach each other to infinitesimal intervals. The boundary orbit itself makes the junction at L_3 and is represented by the asymptote pair in the infinitesimal range. Thuering's theory, which provides only an approximation, also gives $\rho = 0$, for $\dot{\alpha} = 0$ (i.e., the

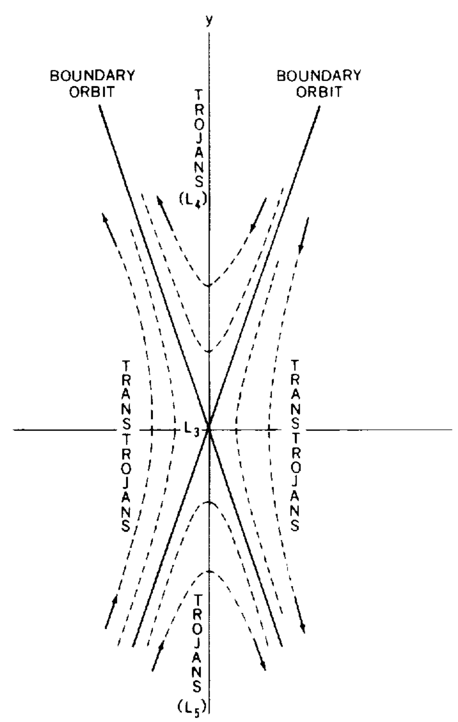

Figure 4—Infinitesimal region around L showing the Trojans, Boundary orbit and Transtrojans.

boundary orbit according to this theory does not make the junction at L_3, but at the opposite point (-1, 0) of Jupiter). The deviation however is of the order m — the order of the terms neglected in Equation 7.

The second group of hyperbolas, whose shape is flat, and which fills the right and left sector of Figure 4, has as common main axis the x-axis itself, and all these hyperbolas thus intersect the x-axis at right angles. They are tangent there to those periodic orbits which jointly enclose L_4 and L_5 and must be considered as continuations of the two groups of Trojan orbits beyond their common boundary orbit. We can use the term "Transtrojans" to describe the planetoids which would run on such horse-shoe-shaped orbits. As a matter of fact, these Transtrojan orbits are among the periodic orbits of the restricted problem — and we can obtain them by numerical integration of this problem's differential equations, if we start from a point on the x-axis which is near L_3 at slow speed perpendicular to the x-axis; and if we vary the speed until the orbit turns back on itself.

Thuering's later attempt to arrive at a periodic orbit starting from the point (-1, 0) when $\dot{y} = 0$, therefore was unsound and necessarily unsuccessful.

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